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Inst. of Computer Science, Univ. of Białystok

Some Basic Properties of Sets

Czesław Byliński
Warsaw University
Białystok

Summary. In this article some basic theorems about singletons, pairs, power sets, unions of families of sets, and the cartesian product of two sets are proved.

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The article [1] provides the notation and terminology for this paper.

1. Cartesian product of sets

In this paper x, x_1, x_2, y, y_1, y_2, z, A, B, X, X_1, X_2, X_3, X_4, Y, Y_1, Y_2, Z, N, M denote sets.

Let us consider X. The functor 2^X is defined by:

(Def. 1) $Z \in 2^X$ iff $Z \subseteq X$.

Let us consider X_1, X_2. The functor $[: X_1, X_2 :]$ is defined as follows:

(Def. 2) $z \in [: X_1, X_2 :]$ iff there exist x, y such that $x \in X_1$ and $y \in X_2$ and $z = \langle x, y \rangle$.

Let us consider X_1, X_2, X_3. The functor $[: X_1, X_2, X_3 :]$ is defined by:

(Def. 3) $[: X_1, X_2, X_3 :] = [: [: X_1, X_2 :], X_3 :]$.

Let us consider X_1, X_2, X_3, X_4. The functor $[: X_1, X_2, X_3, X_4 :]$ is defined as follows:

(Def. 4) $[: X_1, X_2, X_3, X_4 :] = [: [: X_1, X_2, X_3 :], X_4 :]$.

2. Basic properties of sets

We now state a number of propositions:

(1) $2^{\emptyset} = \{\emptyset\}$.

(2) $\bigcup \emptyset = \emptyset$.

(6)[1] If $\{x\} = \{y\}$, then $x = y$.

(8)[2] If $\{x\} = \{y_1, y_2\}$, then $x = y_1$.

(9) If $\{x\} = \{y_1, y_2\}$, then $y_1 = y_2$.

[1] The propositions (3)–(5) have been removed.
[2] The proposition (7) has been removed.

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(10) If $\{x_1, x_2\} = \{y_1, y_2\}$, then $x_1 = y_1$ or $x_1 = y_2$.

(12)[3] $\{x\} \subseteq \{x, y\}$.

(13) If $\{x\} \cup \{y\} = \{x\}$, then $x = y$.

(14) $\{x\} \cup \{x, y\} = \{x, y\}$.

(16)[4] If $\{x\}$ misses $\{y\}$, then $x \neq y$.

(17) If $x \neq y$, then $\{x\}$ misses $\{y\}$.

(18) If $\{x\} \cap \{y\} = \{x\}$, then $x = y$.

(19) $\{x\} \cap \{x, y\} = \{x\}$.

(20) $\{x\} \setminus \{y\} = \{x\}$ iff $x \neq y$.

(21) If $\{x\} \setminus \{y\} = \emptyset$, then $x = y$.

(22) $\{x\} \setminus \{x, y\} = \emptyset$.

(23) If $x \neq y$, then $\{x, y\} \setminus \{y\} = \{x\}$.

(24) If $\{x\} \subseteq \{y\}$, then $x = y$.

(25) If $\{z\} \subseteq \{x, y\}$, then $z = x$ or $z = y$.

(26) If $\{x, y\} \subseteq \{z\}$, then $x = z$.

(27) If $\{x, y\} \subseteq \{z\}$, then $\{x, y\} = \{z\}$.

(28) If $\{x_1, x_2\} \subseteq \{y_1, y_2\}$, then $x_1 = y_1$ or $x_1 = y_2$.

(29) If $x \neq y$, then $\{x\} \dot{-} \{y\} = \{x, y\}$.

(30) $2^{\{x\}} = \{\emptyset, \{x\}\}$.

(31) $\bigcup\{x\} = x$.

(32) $\bigcup\{\{x\}, \{y\}\} = \{x, y\}$.

(33) If $\langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle$, then $x_1 = y_1$ and $x_2 = y_2$.

(34) $\langle x, y \rangle \in [: \{x_1\}, \{y_1\} :]$ iff $x = x_1$ and $y = y_1$.

(35) $[: \{x\}, \{y\} :] = \{\langle x, y \rangle\}$.

(36) $[: \{x\}, \{y, z\} :] = \{\langle x, y \rangle, \langle x, z \rangle\}$ and $[: \{x, y\}, \{z\} :] = \{\langle x, z \rangle, \langle y, z \rangle\}$.

(37) $\{x\} \subseteq X$ iff $x \in X$.

(38) $\{x_1, x_2\} \subseteq Z$ iff $x_1 \in Z$ and $x_2 \in Z$.

(39) $Y \subseteq \{x\}$ iff $Y = \emptyset$ or $Y = \{x\}$.

(40) If $Y \subseteq X$ and $x \notin Y$, then $Y \subseteq X \setminus \{x\}$.

(41) If $X \neq \{x\}$ and $X \neq \emptyset$, then there exists y such that $y \in X$ and $y \neq x$.

(42) $Z \subseteq \{x_1, x_2\}$ iff $Z = \emptyset$ or $Z = \{x_1\}$ or $Z = \{x_2\}$ or $Z = \{x_1, x_2\}$.

(43) If $\{z\} = X \cup Y$, then $X = \{z\}$ and $Y = \{z\}$ or $X = \emptyset$ and $Y = \{z\}$ or $X = \{z\}$ and $Y = \emptyset$.

(44) If $\{z\} = X \cup Y$ and $X \neq Y$, then $X = \emptyset$ or $Y = \emptyset$.

[3] The proposition (11) has been removed.
[4] The proposition (15) has been removed.

(45) If $\{x\} \cup X \subseteq X$, then $x \in X$.

(46) If $x \in X$, then $\{x\} \cup X = X$.

(47) If $\{x,y\} \cup Z \subseteq Z$, then $x \in Z$.

(48) If $x \in Z$ and $y \in Z$, then $\{x,y\} \cup Z = Z$.

(49) $\{x\} \cup X \neq \emptyset$.

(50) $\{x,y\} \cup X \neq \emptyset$.

(51) If $X \cap \{x\} = \{x\}$, then $x \in X$.

(52) If $x \in X$, then $X \cap \{x\} = \{x\}$.

(53) If $x \in Z$ and $y \in Z$, then $\{x,y\} \cap Z = \{x,y\}$.

(54) If $\{x\}$ misses X, then $x \notin X$.

(55) If $\{x,y\}$ misses Z, then $x \notin Z$.

(56) If $x \notin X$, then $\{x\}$ misses X.

(57) If $x \notin Z$ and $y \notin Z$, then $\{x,y\}$ misses Z.

(58) $\{x\}$ misses X or $\{x\} \cap X = \{x\}$.

(59) If $\{x,y\} \cap X = \{x\}$, then $y \notin X$ or $x = y$.

(60) If $x \in X$ and if $y \notin X$ or $x = y$, then $\{x,y\} \cap X = \{x\}$.

(63)[5] If $\{x,y\} \cap X = \{x,y\}$, then $x \in X$.

(64) $z \in X \setminus \{x\}$ iff $z \in X$ and $z \neq x$.

(65) $X \setminus \{x\} = X$ iff $x \notin X$.

(66) If $X \setminus \{x\} = \emptyset$, then $X = \emptyset$ or $X = \{x\}$.

(67) $\{x\} \setminus X = \{x\}$ iff $x \notin X$.

(68) $\{x\} \setminus X = \emptyset$ iff $x \in X$.

(69) $\{x\} \setminus X = \emptyset$ or $\{x\} \setminus X = \{x\}$.

(70) $\{x,y\} \setminus X = \{x\}$ iff $x \notin X$ but $y \in X$ or $x = y$.

(72)[6] $\{x,y\} \setminus X = \{x,y\}$ iff $x \notin X$ and $y \notin X$.

(73) $\{x,y\} \setminus X = \emptyset$ iff $x \in X$ and $y \in X$.

(74) $\{x,y\} \setminus X = \emptyset$ or $\{x,y\} \setminus X = \{x\}$ or $\{x,y\} \setminus X = \{y\}$ or $\{x,y\} \setminus X = \{x,y\}$.

(75) $X \setminus \{x,y\} = \emptyset$ iff $X = \emptyset$ or $X = \{x\}$ or $X = \{y\}$ or $X = \{x,y\}$.

(79)[7] If $A \subseteq B$, then $2^A \subseteq 2^B$.

(80) $\{A\} \subseteq 2^A$.

(81) $2^A \cup 2^B \subseteq 2^{A \cup B}$.

(82) If $2^A \cup 2^B = 2^{A \cup B}$, then A and B are $\subseteq$-comparable.

(83) $2^{A \cap B} = 2^A \cap 2^B$.

(84) $2^{A \setminus B} \subseteq \{\emptyset\} \cup (2^A \setminus 2^B)$.

(86)[8] $2^{A \setminus B} \cup 2^{B \setminus A} \subseteq 2^{A \dot{-} B}$.

[5] The propositions (61) and (62) have been removed.
[6] The proposition (71) has been removed.
[7] The propositions (76)–(78) have been removed.
[8] The proposition (85) has been removed.

(92)[9] If $X \in A$, then $X \subseteq \bigcup A$.

(93) $\bigcup\{X, Y\} = X \cup Y$.

(94) If for every X such that $X \in A$ holds $X \subseteq Z$, then $\bigcup A \subseteq Z$.

(95) If $A \subseteq B$, then $\bigcup A \subseteq \bigcup B$.

(96) $\bigcup(A \cup B) = \bigcup A \cup \bigcup B$.

(97) $\bigcup(A \cap B) \subseteq \bigcup A \cap \bigcup B$.

(98) If for every X such that $X \in A$ holds X misses B, then $\bigcup A$ misses B.

(99) $\bigcup(2^A) = A$.

(100) $A \subseteq 2^{\bigcup A}$.

(101) If for all X, Y such that $X \neq Y$ and $X \in A \cup B$ and $Y \in A \cup B$ holds X misses Y, then $\bigcup(A \cap B) = \bigcup A \cap \bigcup B$.

(102) If $z \in [: X, Y :]$, then there exist x, y such that $\langle x, y \rangle = z$.

(103) If $A \subseteq [: X, Y :]$ and $z \in A$, then there exist x, y such that $x \in X$ and $y \in Y$ and $z = \langle x, y \rangle$.

(104) If $z \in [: X_1, Y_1 :] \cap [: X_2, Y_2 :]$, then there exist x, y such that $z = \langle x, y \rangle$ and $x \in X_1 \cap X_2$ and $y \in Y_1 \cap Y_2$.

(105) $[: X, Y :] \subseteq 2^{2^{X \cup Y}}$.

(106) $\langle x, y \rangle \in [: X, Y :]$ iff $x \in X$ and $y \in Y$.

(107) If $\langle x, y \rangle \in [: X, Y :]$, then $\langle y, x \rangle \in [: Y, X :]$.

(108) If for all x, y holds $\langle x, y \rangle \in [: X_1, Y_1 :]$ iff $\langle x, y \rangle \in [: X_2, Y_2 :]$, then $[: X_1, Y_1 :] = [: X_2, Y_2 :]$.

(109) If $A \subseteq [: X, Y :]$ and for all x, y such that $\langle x, y \rangle \in A$ holds $\langle x, y \rangle \in B$, then $A \subseteq B$.

(110) If $A \subseteq [: X_1, Y_1 :]$ and $B \subseteq [: X_2, Y_2 :]$ and for all x, y holds $\langle x, y \rangle \in A$ iff $\langle x, y \rangle \in B$, then $A = B$.

(111) If for every z such that $z \in A$ there exist x, y such that $z = \langle x, y \rangle$ and for all x, y such that $\langle x, y \rangle \in A$ holds $\langle x, y \rangle \in B$, then $A \subseteq B$.

(112) Suppose that

(i) for every z such that $z \in A$ there exist x, y such that $z = \langle x, y \rangle$,

(ii) for every z such that $z \in B$ there exist x, y such that $z = \langle x, y \rangle$, and

(iii) for all x, y holds $\langle x, y \rangle \in A$ iff $\langle x, y \rangle \in B$.

Then $A = B$.

(113) $[: X, Y :] = \emptyset$ iff $X = \emptyset$ or $Y = \emptyset$.

(114) If $X \neq \emptyset$ and $Y \neq \emptyset$ and $[: X, Y :] = [: Y, X :]$, then $X = Y$.

(115) If $[: X, X :] = [: Y, Y :]$, then $X = Y$.

(116) If $X \subseteq [: X, X :]$, then $X = \emptyset$.

(117) If $Z \neq \emptyset$ and if $[: X, Z :] \subseteq [: Y, Z :]$ or $[: Z, X :] \subseteq [: Z, Y :]$, then $X \subseteq Y$.

(118) If $X \subseteq Y$, then $[: X, Z :] \subseteq [: Y, Z :]$ and $[: Z, X :] \subseteq [: Z, Y :]$.

(119) If $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$, then $[: X_1, X_2 :] \subseteq [: Y_1, Y_2 :]$.

[9] The propositions (87)–(91) have been removed.

(120) $[:X \cup Y, Z:] = [:X, Z:] \cup [:Y, Z:]$ and $[:Z, X \cup Y:] = [:Z, X:] \cup [:Z, Y:]$.

(121) $[:X_1 \cup X_2, Y_1 \cup Y_2:] = [:X_1, Y_1:] \cup [:X_1, Y_2:] \cup [:X_2, Y_1:] \cup [:X_2, Y_2:]$.

(122) $[:X \cap Y, Z:] = [:X, Z:] \cap [:Y, Z:]$ and $[:Z, X \cap Y:] = [:Z, X:] \cap [:Z, Y:]$.

(123) $[:X_1 \cap X_2, Y_1 \cap Y_2:] = [:X_1, Y_1:] \cap [:X_2, Y_2:]$.

(124) If $A \subseteq X$ and $B \subseteq Y$, then $[:A, Y:] \cap [:X, B:] = [:A, B:]$.

(125) $[:X \setminus Y, Z:] = [:X, Z:] \setminus [:Y, Z:]$ and $[:Z, X \setminus Y:] = [:Z, X:] \setminus [:Z, Y:]$.

(126) $[:X_1, X_2:] \setminus [:Y_1, Y_2:] = [:X_1 \setminus Y_1, X_2:] \cup [:X_1, X_2 \setminus Y_2:]$.

(127) If X_1 misses X_2 or Y_1 misses Y_2, then $[:X_1, Y_1:]$ misses $[:X_2, Y_2:]$.

(128) $\langle x, y \rangle \in [:\{z\}, Y:]$ iff $x = z$ and $y \in Y$.

(129) $\langle x, y \rangle \in [:X, \{z\}:]$ iff $x \in X$ and $y = z$.

(130) If $X \neq \emptyset$, then $[:\{x\}, X:] \neq \emptyset$ and $[:X, \{x\}:] \neq \emptyset$.

(131) If $x \neq y$, then $[:\{x\}, X:]$ misses $[:\{y\}, Y:]$ and $[:X, \{x\}:]$ misses $[:Y, \{y\}:]$.

(132) $[:\{x,y\}, X:] = [:\{x\}, X:] \cup [:\{y\}, X:]$ and $[:X, \{x,y\}:] = [:X, \{x\}:] \cup [:X, \{y\}:]$.

(134)[10] If $X_1 \neq \emptyset$ and $Y_1 \neq \emptyset$ and $[:X_1, Y_1:] = [:X_2, Y_2:]$, then $X_1 = X_2$ and $Y_1 = Y_2$.

(135) If $X \subseteq [:X, Y:]$ or $X \subseteq [:Y, X:]$, then $X = \emptyset$.

(136) There exists M such that

(i) $N \in M$,

(ii) for all X, Y such that $X \in M$ and $Y \subseteq X$ holds $Y \in M$,

(iii) for every X such that $X \in M$ holds $2^X \in M$, and

(iv) for every X such that $X \subseteq M$ holds $X \approx M$ or $X \in M$.

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[10] The proposition (133) has been removed.